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A Limiting Absorption Principle for Schrödinger Operators with Generalized Von Neumann–Wigner Potentials II. The Proof

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DEDICATED TO PROFESSOR HUGH TURITTIN ON HIS 90TH BIRTHDAY.

In this series of papers we prove the limiting absorption principle over a given interval for a class of Hamiltonians which contains the original one of von Neumann and Wigner. More specifically, the Hamiltonians are of the form $-\Delta + c \sin b|x|/|x|^{\beta} + V(x)$, where $2/3 < \beta \leq 1$, $V(x)$ is a short range potential and $\mathscr{I}$ is a given compact subinterval of the open positive axis which does not contain the point $b^2/4$.

INTRODUCTION

In this Part II of this series of papers we prove Theorem 3.1 which was stated in Part I. We continue the numbering of sections, however, we start anew the numbering of references.

In Section 7 we complete the proof of Theorem 3.1 under the additional condition (3.4). In Proposition 7.1 we formulate a lower estimate for any solution of the basic equation (2.6) over any family of intervals $\mathscr{I}_{m1}$ satisfying assumption (3.5). This is done by applying the general Proposition 4.1 to such intervals. Similarly, in Proposition 7.2 we formulate lower

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estimates for any family of intervals $\mathscr{I}_r$ satisfying assumption (3.7). Finally, combining Propositions 7.1 and 7.2 we arrive at conclusion (3.12) of Theorem 3.1. This completes the proof of Theorem 3.1.

In Section 8 we verify the approximate phase assumptions on $\mathscr{I}_r$ without the additional assumption (3.4). For this purpose, we introduce a family of subintervals $\mathscr{I}_{m2}$ of the family of intervals $\mathscr{I}_r$ satisfying assumption (3.7). Then we redefine the approximate phase on the family of intervals $\mathscr{I}_{m2}$. To verify assumptions (4.7) and (4.6) on the complements, $\mathscr{I}_r \setminus \mathscr{I}_{m2}$, we formulate Lemma 8.1. Similarly to Section 6, the main difficulty is to verify assumption (4.8) and the validity of this assumption is the statement of Theorem 8.2. Note that conclusion (6.9) of Theorem 6.2 with the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$ in place of the intervals $\mathscr{I}_r$ yields conclusion (8.16) of Theorem 8.2. We start the proof of Theorem 8.2 with Proposition 8.3. This proposition implies via the algebraic Lemma 6.3 that Theorem 8.2 holds under the more restricted assumption $3/4 \leq \beta \leq 1$. We prove Proposition 8.3 by repeated applications of Lemma 8.4. This lemma is an adaptation of Lemma 6.5 to these complements. More specifically, it is an adaptation of the scaling Lemma 5.3 to the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$ combined with Lemma 8.1. We complete the proof of Theorem 8.2 with Proposition 8.5. This proposition sharpens the conclusions of Proposition 8.3 so that they imply Theorem 8.2 under the original assumption of Theorem 2.1 that $2/3 < \beta \leq 1$. We prove Proposition 8.4 by repeated applications of Lemma 8.6, which is a sharper version of Lemma 8.4. Hence Lemma 8.6 is a sharper adaptation of Lemma 6.5 to the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$. The proof of this adaptation of Lemma 6.5 is more technical than the previous one given in Lemma 8.4 and so, we give the proof of Lemma 8.6 in the Appendix. We conclude this section by verifying the approximate phase assumptions on the subintervals $\mathscr{I}_{m2}$ and we do this using the scaling Lemma 5.3.

In Section 9 we prove Theorem 3.1 without the additional assumption (3.4). This short section is, essentially, a combination of Sections 7 and 8.

For the related question of Mourre estimates we refer to the book of Cycon, Froese, Kirsch and Simon [CFKS]. For the related property of absolute continuity for ordinary differential operators we refer to the many excellent research papers.

7. THE PROOF OF THEOREM 3.1 WITH THE ADDITIONAL ASSUMPTION (3.4)

As a first step of the proof of Theorem 3.1, we formulate a version of conclusion (3.12) which involves only intervals $\mathscr{I}_{m1}$ and the approximate phases θ_{m1} over them.

PROPOSITION 7.1. *Let the assumptions of Theorem* 3.1 *hold. Then, there are constants* $\gamma > 0$ *and* ν_0 *such that for each interval* $\mathscr{I}_{m1}$ *satisfying assumption* (3.5) *and for each* $\lambda \in \mathscr{I}$

$$F(\theta_m, f)(\sup \mathscr{I}_{m1}) \geq \gamma \cdot \nu^{(2\beta-1-\kappa)/2} \cdot F(\theta_m, f)(\inf \mathscr{I}_{m1}), \qquad \textit{for } \nu > \nu_0. \tag{7.1}$$

Theorem 5.1 and the remarks before it show that the approximate phase of definition (3.6) satisfies the assumptions of the general Proposition 4.1 over any interval $\mathscr{I}_{m1}$ which satisfies assumption (3.5). Then conclusion (4.9) with $\mathscr{I}_{m1}$ in place of $\mathscr{I}$, θ_m in place of θ and $\sup \mathscr{I}_{m1}$ in place of ρ yields the lower estimate

$$\begin{aligned} &F(\theta_m, f)(\sup \mathscr{I}_{m1}) \\ &\qquad \geq \gamma \cdot \exp\left[2\int_{\inf \mathscr{I}_{m1}}^{\sup \mathscr{I}_{m1}} \operatorname{Im}{}_m^{\theta}(\sigma)\, d\sigma\right] \cdot F(\theta_m, f)(\inf \mathscr{I}_{m1}). \end{aligned} \tag{7.2}$$

The lower estimate (7.2) allows us to complete the proof of Proposition 7.1 by showing that there is a constant γ such that

$$\exp\left[2\int_{\mathscr{I}_{m1}} \operatorname{Im} \theta_m(\sigma)\, d\sigma\right] > \gamma \cdot \nu^{(2\beta-1-\kappa)/2}, \qquad \text{for } \nu > \nu_0. \tag{7.3}$$

To prove the lower estimate (7.3) note relation (5.5) and formula (5.1) together give

$$2 \operatorname{Im} \theta_m = (\lambda - \psi_\nu)^{-1} \nu^{-1} \psi_\nu^{3/2}. \tag{7.4}$$

Next, we apply conclusion (5.10) of the scaling Lemma 5.3 to $\omega_1 = -1$, $\omega_2 = 3/2$ and $\mathscr{I} = \mathscr{I}_{m1}$. Then using assumption (3.5) and that the integrand is positive we find

$$\int_{\mathscr{I}_{m1}} (\lambda - \psi_\nu(\rho))^{-1} \psi_\nu^{2/3}(\rho)\, d\rho \geq \nu \cdot \int_{1+2\nu^{\kappa-1}}^{1+\nu^{2\beta-2}} (\sigma^2 - 1)^{-1} \sigma^{-1}\, d\sigma.$$

An elementary partial fraction decomposition gives

$$\begin{aligned} \int_{1+2\nu^{\kappa-1}}^{1+\nu^{2\beta-2}} (\sigma^2 - 1)^{-1} \sigma^{-1}\, d\sigma = \frac{1}{2}\log \nu^{2\beta-1-\kappa} - \frac{1}{2}\log\frac{2+\nu^{2\beta-2}}{1+\nu^{\kappa-1}} \\ + \log\frac{1+\nu^{2\beta-2}}{1+2\nu^{\kappa-1}}. \end{aligned} \tag{7.5}$$

Now we define

$$\gamma = \frac{1}{2} \cdot \lim_{\nu\to\infty} \exp\left(-\frac{1}{2}\log\frac{2+\nu^{2\beta-2}}{1+\nu^{\kappa-1}} + \log\frac{1+\nu^{2\beta-2}}{1+2\nu^{\kappa-1}}\right).$$

Since we see from assumptions (3.1) and (2.2) that $2\beta - 2 \leq 0$ and $\kappa - 1 \leq 0$, the previous constant γ is strictly positive. At the same time it follows that the lower estimate (7.3) holds for this constant.

Finally, inserting the lower estimate (7.3) into the lower estimate (7.2) we arrive at conclusion (7.1) of Proposition 7.1.

As a second step of the proof of Theorem 3.1, we formulate a version of conclusion (3.12) which involves only intervals $\mathscr{J}_r$ and the approximate phases θ_r over them.

PROPOSITION 7.2. *Let the assumptions of Theorem* 3.1 *hold and let the conclusions of Theorem* 6.2 *and Lemma* 6.1 *hold for the family intervals* $\mathscr{J}_r$. *Then, to each* $\delta > 1$ *there are constants* $\gamma > 0$ *and* ν_0 *such that for each interval* $\mathscr{J}_r$ *and for each* $\lambda \in \mathscr{I}$

$$F(\theta_r, f)(\rho) \geq \gamma \cdot \nu^{1-\beta} \cdot F(\theta_r, f)(\inf \mathscr{J}_r), \quad \nu > \nu_0, \rho \in \mathscr{J}_r, \rho \geq \delta\lambda^{-1/2}\nu. \tag{7.6}$$

According to Section 6 the approximate phase of definition (3.10) satisfies the assumptions of the general Proposition 4.1 over any intervals $\mathscr{J}_r$ which satisfy assumption (3.5). Then conclusion (4.9) with $\mathscr{J}_r$ in place of $\mathscr{J}$, θ_r in place of θ and $\sup \mathscr{J}_r$ in place of ρ yields the lower estimate

$$F(\theta_r, f)(\rho) \geq \gamma \cdot \exp\left[2\int_{\inf \mathscr{J}_r}^{\rho} \operatorname{Im} \theta_r(\sigma)\, d\sigma\right] \cdot F(\theta_r, f)(\inf \mathscr{J}_r),$$
$$\nu > \nu_0, \rho \in \mathscr{J}_r, \rho \geq \delta\lambda^{-1/2}\nu. \tag{7.7}$$

The lower estimate (7.7) allows us to complete the proof of Proposition 7.2 by showing that to the constant $\delta > 1$ there are constants γ and ν_0 such that

$$\exp\left[2\int_{\mathrm{nf}\, \mathscr{J}_r}^{\rho} \operatorname{Im} \theta_r(\sigma)\, d\sigma\right] > \gamma\nu^{1-\beta}, \qquad \text{for } \nu > \nu_0 \text{ and } \rho \geq \delta\lambda^{-1/2}\nu. \tag{7.8}$$

We start the proof of the lower estimate (7.8) by showing that

$$\exp\left[2\int_{\inf \mathscr{J}_r}^{\rho} \operatorname{Im} \theta_m(\sigma)\, d\sigma\right] > \gamma\nu^{1-\beta}, \qquad \text{for } \nu > \nu_0 \text{ and } \rho \geq \delta\lambda^{-1/2}\nu. \tag{7.9}$$

To prove this lower estimate, we apply conclusion (5.10) of the scaling Lemma 5.3 with $\omega_1 = -1$, $\omega_2 = 3/2$ and $\mathscr{J} = (\inf \mathscr{J}_r, \rho)$. Then using assumption (3.7) and the fact that the integrand is positive, we find

$$\int_{\inf \mathscr{J}_r}^{\rho} (\lambda - \psi_\nu(\sigma))^{-1} \psi_\nu^{3/2}(\sigma)\, d\sigma \geq \nu \cdot \int_{1+2\nu^{2\beta-2}}^{\delta} (\sigma^2 - 1)^{-1}\, d\sigma,$$

$$\text{for } \rho \geq \delta \lambda^{-1/2} \nu.$$

An elementary partial fraction decomposition gives,

$$\int_{1+2\nu^{2\beta-2}}^{\delta} (\sigma^2 - 1)^{-1} \sigma^{-1}\, d\sigma = \frac{1}{2} \log \nu^{2-2\beta} + \frac{1}{2} \log\left(\frac{\delta - 1}{\delta + 1} (1 + \nu^{2\beta - 2}) \right)$$
$$+ \log\left(\frac{\delta}{1 + 2\nu^{2\beta-2}} \right). \tag{7.10}$$

Since by assumption $\delta > 1$, the right side of formula (7.10) is real. This fact allows us to repeat the proof of the lower estimate (7.3) with formula (7.10) in place of formula (7.5) and so, the lower estimate (7.9) follows.

We complete the proof of the lower estimate (7.8) by showing the key estimate:

$$\limsup_{\nu \to \infty} \int_{\mathscr{J}_r} |2 \operatorname{Im} \theta_r - 2 \operatorname{Im} \theta_m |(\rho)\, d\rho < \infty. \tag{7.11}$$

As a first step of the proof of the key estimate (7.11) we combine definitions (3.10), (3.8) with formula (5.1) and with relation (6.1). This yields

$$2 \operatorname{Im} \theta_r - 2 \operatorname{Im} \theta_m = \frac{1}{2} \left[(\lambda - \psi_\nu)^{-1} - (\lambda - \psi_r)^{-1} \right] \psi_\nu'$$
$$- \frac{1}{2} (\lambda - \psi_r)^{-1} [\psi_r' - \psi_\nu']. \tag{7.12}$$

As a second step of the proof of the key estimate (7.11) we show that the integral of the absolute value of the first term of formula (7.12) is bounded. Specifically, we show that

$$\int_{\mathscr{J}_r} \left| \left[(\lambda - \psi_\nu)^{-1} - (\lambda - \psi_r)^{-1} \right] \psi_\nu' \right| (\rho)\, d\rho$$
$$= O\left(\nu^{-\beta} (1 + |\log \nu^{2-2\beta}|) \right), \qquad \nu \to \infty. \tag{7.13}$$

To prove estimate (7.13) we observe that definition (3.9) yields

$$\left[(\lambda - \psi_r)^{-1} - (\lambda - \psi_\nu)^{-1} \right] = -(\lambda - \psi_r)^{-1} (\lambda - \psi_\nu)^{-1} a_r p_0, \tag{7.14}$$

and so, multiplying formula (7.14) by ψ_ν' and using formula (5.5), we find

$$\left[(\lambda - \psi_\nu)^{-1} - (\lambda - \psi_r)^{-1}\right]\psi_\nu' = 2\nu^{-1}\psi_\nu^{3/2}(\lambda - \psi_r)^{-1}(\lambda - \psi_\nu)a^{-1}a_r p_0. \tag{7.15}$$

Integrating the absolute value of estimate (7.15) over the interval $\mathscr{J}_r$ and using conclusion (6.3) of Lemma 6.1 and estimate (2.15) we obtain

$$\int_{\mathscr{J}_r}\left|\left[(\lambda - \psi_\nu)^{-1} - (\lambda - \psi_r)^{-1}\right]\psi_\nu'\right|(\rho)\,d\rho$$
$$= O(\nu^{-1-\beta})\cdot\int_{\mathscr{J}_r}\left|(\lambda - \psi_r)^{-1}\psi_\nu^{3/2+\beta}a_r\right|(\rho)\,d\rho. \tag{7.16}$$

We see from conclusion (6.20) of Lemma 6.5 that

$$\sup_{\lambda\in\mathscr{J}}\int_{\mathscr{J}_r}\left|(\lambda - \psi_\nu)^{\omega_1}\psi_\nu^{\omega_2}a_r^{\omega_3}\right|(\rho)\,d\rho = O\big(\nu(1 + |\log\nu^{2-2\beta}|\big),$$
$$\omega_1 + \omega_3 \geq -1,\ \omega_2 > \frac{1}{2},\ \nu\to\infty. \tag{7.17}$$

Combining estimates (7.17) and (7.16) we obtain estimate (7.13).

As a third and final step of the proof the key estimate (7.11) we show that the absolute value of the integral of the second term of formula (7.12) is bounded. Specifically, we show that

$$\left|\int_{\mathscr{J}_r}(\lambda - \psi_r)^{-1}(a_r p_0)'(\rho)\,d\rho\right| = O\big(\nu^{1-2\beta}(1 + |\log\nu^{2-2\beta}|)\big),$$
$$\text{for } \nu\to\infty. \tag{7.18}$$

To prove estimate (7.18) we integrate by parts,

$$\int_{\mathscr{J}_r}(\lambda - \psi_r)^{-1}(a_r p_0)'(\rho)\,d\rho = \left[(\lambda - \psi_r)^{-1}a_r p_0\right](\partial\mathscr{J}_r)$$
$$-\int_{\mathscr{J}_r}(\lambda - \psi_r)^{-2}\psi_r' a_r p_0(\rho)\,d\rho. \tag{7.19}$$

Combining conclusions (6.3) and (6.2) of Lemma 6.1 with estimate (2.15) we find

$$\left|\left[(\lambda - \psi_r)^{-1}a_r p_0\right](\partial\mathscr{J}_r)\right| = O(\nu^{-\beta}), \qquad \text{for } \nu\to\infty. \tag{7.20}$$

Next we show that

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_r} \left|(\lambda - \psi_r)^{-2} \psi_r' a_r p_0\right|(\rho)\, d\rho = O\left(\nu^{1-2\beta}\left(1 + |\log \nu^{2-2\beta}|\right)\right),$$

$$\text{for } \nu \to \infty. \quad (7.21)$$

To see estimate (7.21) we differentiate definition (3.9), use formula (5.5) and multiply the resulting formula by the absolute value of $a_r p_0$. This leads to the inequality

$$|\psi_r' a_r p_0| \le 2\nu^{-1}\psi_\nu^{3/2}|a_r p_0| + |p_0' a_r^2 p_0| + |a_r' a_r p_0^2|. \quad (7.22)$$

To estimate ν times the integral corresponding to the first term on the right of inequality (7.22) we apply estimate (7.17) to $\omega_1 = -2$, $\omega_2 = (3+\beta)/2$, $\omega_3 = 1$. Then using estimate (2.15) we find

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_r} \left|(\lambda - \psi_r)^{-2} \psi_\nu^{3/2} a_r p_0\right|(\rho)\, d\rho = O\left(\nu^{1-\beta}\left(1 + |\log \nu^{2-2\beta}|\right)\right),$$

$$\text{for } \nu \to \infty. \quad (7.23)$$

To estimate the integral corresponding to second term on the right of inequality (7.22) we apply estimate (7.17) to $\omega_1 = -2$, $\omega_2 = \beta$, $\omega_3 = 2$. Then using estimate (2.15) we find

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_r} \left|(\lambda - \psi_r)^{-2} a_r^2 p_0' p_0\right|(\rho)\, d\rho = O\left(\nu^{1-2\beta}\left(1 + |\log \nu^{2-2\beta}|\right)\right),$$

$$\text{for } \nu \to \infty. \quad (7.24)$$

To estimate the integral corresponding to the third term on right of inequality (7.22) we combine formula (6.29) with estimate (2.15). Then using conclusion (6.3) of Lemma 6.1 we find

$$\int_{\mathscr{I}_r} \left|(\lambda - \psi_r)^{-2} a_r' a_r^{1+2} p_0^2\right|(\rho)\, d\rho$$

$$= O(\nu^{-1-2\beta}) \cdot \sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_r} \left|(\lambda - \psi_\nu)^{-2-2} \psi_\nu^{3/2+\beta} a_r^3\right|(\rho)\, d\rho.$$

Application of estimate (7.17) to $\omega_1 = -2-2$, $\omega_2 = 3/2+\beta$, $\omega_3 = 1 + 2$ shows that the second factor on the right is of the order of $\nu(1 + |\log \nu^{2-2\beta}|)$ and so

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_r} \left|(\lambda - \psi_r)^{-2} a_r' a_r p_0^2\right|(\rho)\, d\rho = O\left(\nu^{-2\beta}\left(1 + |\log \nu^{2-2\beta}|\right)\right),$$

$$\text{for } \nu \to \infty. \quad (7.25)$$

Finally we multiply inequality (7.22) by $(\lambda - \psi_r)^{-2}$ and integrate the resulting inequality over the intervals $\mathscr{I}_r$. Then using estimates (7.25), (7.24), and (7.23) we obtain estimate (7.21) via the triangle inequality. Then combining estimates (7.21) and (7.20) we find estimate (7.18). Thus, the key estimate (7.11) follows.

As a third and final step of the proof of Theorem 3.1 we show that under the additional assumption (3.4), Propositions 7.1 and 7.2 imply it. To see this, note that according to definition (2.1) in each closed interval of length at least $2\pi/b$ the potential p_0 has at least one zero. This fact allows us to choose an interval $\mathscr{I}_{m1}$ which satisfies assumption (3.5) and is such that $p_0(\sup \mathscr{I}_{m1}) = 0$. Then we see from definitions (3.10), (3.9), and (3.6) that

$$\theta_r(\sup \mathscr{I}_{m1}) = \theta_m(\sup \mathscr{I}_{m1}) - i\frac{1}{4}(\lambda - \psi_\nu(\sup \mathscr{I}_{m1}))^{-1}(a_r p_0)'(\sup \mathscr{I}_{m1}). \tag{7.26}$$

Next we claim that

$$\theta_r(\sup \mathscr{I}_{m1}) \sim \theta_m(\sup \mathscr{I}_{m1}), \qquad \text{for } \nu \to \infty. \tag{7.27}$$

Relation (7.26), conclusion (6.2) of Lemma 6.1, formula (6.29) and estimate (2.15) together show that the asymptotic formula (7.27) is implied by

$$\lim_{\nu \to \infty} \nu^{-\beta} \cdot \left|(\lambda - \psi_\nu)^{-1/2}\right|(\sup \mathscr{I}_{m1}) = 0. \tag{7.28}$$

To see estimate (7.28) we apply the elementary identity

$$a^2 - b^{-2} = [ab - 1] \cdot b^{-1}[a + b^{-1}],$$

to $a = \lambda^{1/2}$, $b = \nu^{-1}\rho$. Then using definition (2.14) we find

$$\lambda - \psi_\nu(\rho) = \left[\lambda^{1/2}\nu^{-1}\rho - 1\right] \cdot \nu\rho^{-1}\left[\lambda^{1/2} + \nu\rho^{-1}\right]. \tag{7.29}$$

Applying identity (7.29), in turn, to $\sup \mathscr{I}_{m1}$ in place of ρ and using assumption (3.5) and the fact that the function $\lambda - \psi_\nu$ is increasing, we obtain

$$(\lambda - \psi_\nu)(\sup \mathscr{I}_{m1}) = [\nu^{2\beta-2}] \cdot \lambda(1 + \nu^{2\beta-2})^{-1}\left[1 + (1 + \nu^{2\beta-2})^{-1}\right].$$

We see from the previous estimate that the second factor on the left of estimate (7.28) is of the order of $\nu^{1-\beta}$ and so, estimate (7.28) follows and so does the asymptotic formula (7.27). Thus the definition

$$\mathscr{I}_r = (\sup \mathscr{I}_{m1}, \infty) \tag{7.30}$$

gives an interval for which

$$F(\theta_r, f)(\inf \mathscr{J}_r) \sim F(\theta_m, f)(\sup \mathscr{J}_{m1}), \qquad \text{for } \nu \to \infty. \tag{7.31}$$

Inserting the asymptotic formula (7.31) into conclusion (7.1) of Proposition 7.1 we find, with a possibly different constant γ the lower estimate

$$F(\theta_r, f)(\inf \mathscr{J}_r) \geq \gamma \cdot \nu^{(2\beta-1-\kappa)/2} \cdot F(\theta_m, f)(\inf \mathscr{J}_{m1}), \qquad \text{for } \nu > \nu_0. \tag{7.32}$$

Since the intervals $\mathscr{J}_{m1}$ satisfy assumption (3.5) we see from definition (7.30) that the intervals $\mathscr{J}_r$ satisfy assumption (3.7). Hence we can apply conclusion (7.6) of Proposition 7.2 to them. Then using the lower estimate (7.32) we obtain, with a possibly different constant γ,

$$F(\theta_r, f)(\rho) \geq \gamma \cdot \nu^{1-\beta} \cdot \nu^{(2\beta-1-\kappa)/2} \cdot F(\theta_m, f)(\inf \mathscr{J}_{m1}),$$
$$\text{for } \nu > \nu_0 \text{ and } \rho \geq \delta\lambda^{-1/2}\nu. \tag{7.33}$$

Thus, conclusion (3.12) of Theorem 3.1 follows.

8. VERIFICATION OF THE APPROXIMATE PHASE ASSUMPTIONS ON $\mathscr{J}_r$ WITHOUT (3.4)

In this section we first redefine the function θ_r and then show that this redefined θ_r satisfies the approximate phase assumptions without the additional assumption (3.4). Note that according to assumption (2.11) of Theorem 2.2, in this case we have the other additional assumption,

$$\mathscr{I} \subset \left(\frac{b^2}{4}, \infty\right) \qquad \text{and} \qquad \mathscr{I} \text{ is compact}. \tag{8.1}$$

Then we can no longer invoke conclusions (6.3) and (6.2) of Lemma 6.1 to prove that the approximate phase θ_r of definition (3.10) satisfies the assumptions of the general Proposition 4.1 over the entire interval $\mathscr{J}_r$, as we have done it in Sections 6 and 7. In fact, we claim that in this case for some value of the parameter λ the function θ_r has a pole in $\mathscr{J}_r$. To see this, for each b define the positive number,

$$\lambda_b = \lambda - \frac{b^2}{4}. \tag{8.2}$$

Then we see from formula (6.6) and from definitions (6.4) and (2.14) that the function a_r has a pole at the point $\lambda_b^{-1/2}\nu$. Combination of this fact with definitions (3.10) and (3.9) yields our claim.

To treat this difficulty we define a family of "middle"-intervals centered around this pole. More specifically, first we use the additional assumption (8.1) and definition (8.2) to choose a number d so that

$$0 < \sup_{\lambda \in \mathscr{I}} d_\lambda < d < 1, \qquad \text{where } d_\lambda = \lambda_b^{1/2} \lambda^{-1/2}. \tag{8.3}$$

Second, we define

$$\mathscr{I}_{m2} = \lambda_b^{-1/2} \nu \cdot \left(1 - \frac{1-d}{2} \nu^{\beta - 1}, 1 + 2\nu^{\beta-1} \right). \tag{8.4}$$

Third, similarly to the case of the "middle"-intervals $\mathscr{I}_{m1}$ of assumption (3.5), on the previous "middle"-intervals we define our approximate phase by formula (3.6). In other words, we set

$$\theta_m = (\lambda - \psi_\nu)^{1/2} - i\frac{1}{4}(\lambda - \psi_\nu)^{-1} \psi_\nu' \text{ on } \mathscr{I}_{m2}, \tag{8.5}$$

and keep definition (3.10) on the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$.

Similarly to Section 6, we show that θ_r satisfies assumptions (4.7) and (4.6) over the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$ by showing that conclusion (6.3) of Lemma 6.1 holds for these complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$ in place of the intervals $\mathscr{I}_r$. This is conclusion (8.7) of the following Lemma 8.1. We use conclusion (8.6) to prove conclusion (8.7) as well as to verify assumption (4.8). Note that conclusion (8.6) is a weaker version of conclusion (6.2) of Lemma 6.1 inasmuch as its bound is not uniform in ν.

LEMMA 8.1. *Let the assumptions and notations of Lemma* 6.1 *hold and instead of the additional assumption* (3.4) *let the additional assumption* (8.1) *hold. Next let the intervals* $\mathscr{I}_{m2}$ *be given by definition* (8.4). *Then*

$$\sup_{\lambda \in \mathscr{I}} \sup_{\rho \in \mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_\nu)^{-1} a_r \right| (\rho) = O(\nu^{1-\beta}), \qquad \text{for } \nu \to \infty. \tag{8.6}$$

and

$$\lim_{\nu \to \infty} \sup_{\rho \in \mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_\nu)^{-1} (\lambda - \psi_r) - 1 \right| (\rho) = 0. \tag{8.7}$$

To prove conclusion (8.6) we combine definitions (8.2) and (6.4) with formula (6.6). This yields

$$(\lambda - \psi_\nu)^{-1} a_r = (\lambda_b - \psi_\nu)^{-1}, \tag{8.8}$$

and so, conclusion (8.6) is equivalent to

$$\liminf_{\nu \to \infty} \inf_{\lambda \in \mathscr{I}} \inf_{\rho \in \mathscr{I}_r \setminus \mathscr{I}_{m2}} \nu^{1-\beta} \left| (\lambda_b - \psi_\nu(\rho)) \right| > 0. \tag{8.9}$$

To see estimate (8.9) we claim that

$$\liminf_{\nu\to\infty} \inf_{\lambda\in\mathscr{I}} \nu^{1-\beta}(\lambda_b - \psi_\nu)(\sup \mathscr{I}_{m2}) > 0. \tag{8.10}$$

Indeed, formula (7.29) with λ_b in place of λ and $\sup \mathscr{I}_{m2}$ in place ρ and the use of definition (8.4) yields

$$\lambda_b - \psi_\nu(\sup \mathscr{I}_{m2}) = [2\nu^{\beta-1}] \cdot \lambda_b(1 + 2\nu^{\beta-1})^{-1}\Big[1 + (1 + 2\nu^{\beta-1})^{-1}\Big]. \tag{8.11}$$

We see from definition (8.2) and from the additional assumption (8.1) that the second factor on the right of formula (8.11) is positive and it remains bounded away from zero for $\lambda \in \mathscr{I}$. Hence, formula (8.11) gives estimate (8.10). Similarly to formula (8.11) we see that

$$\begin{aligned}\lambda_b - \psi_\nu(\inf \mathscr{I}_{m2}) &= \left[-\frac{1-d}{2}\nu^{\beta-1}\right] \cdot \lambda_b\left(1 - \frac{1-d}{2}\nu^{\beta-1}\right)^{-1} \\ &\quad\times\left[1 + \left(1 - \frac{1-d}{2}2\nu^{\beta-1}\right)^{-1}\right]\end{aligned} \tag{8.12}$$

and similarly to estimate (8.10) we see that

$$\limsup_{\nu\to\infty} \sup_{\lambda\in\mathscr{I}} \nu^{1-\beta}(\lambda_b - \psi_\nu)(\inf \mathscr{I}_{m2}) < 0. \tag{8.13}$$

Since the function $(\lambda_b - \psi_\nu)$ is monotone increasing, we see from definition (2.14) and from estimate (8.13) that it is negative to the left of the point $\inf \mathscr{I}_{m2}$. Similarly we see from definition (8.2), from the additional assumption (8.1) and from estimate (8.10) that it is positive to the right of the point $\sup \mathscr{I}_{m2}$. Thus combining estimates (8.10) and (8.13) we obtain estimate (8.9) and hence conclusion (8.6).

To prove conclusion (8.7) we combine estimate (2.15) with the already established conclusion (8.6). This yields

$$\lim_{\nu\to\infty} \sup_{\rho\in\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda - \psi_\nu)^{-1} a_r p_0\right|(\rho) = 0. \tag{8.14}$$

Combining estimate (8.14), in turn, with formula (6.7) we obtain conclusion (8.7). This completes the proof of Lemma 8.1.

Next we show that the function θ_r of definition (3.10) satisfies the remaining approximate phase assumption (4.8) over the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$. This is the statement of the theorem that follows.

THEOREM 8.2. *Let the intervals* $\mathscr{J}_r$ *satisfy assumption* (3.8) *and let the approximate phase* θ_r *be given by definition* (3.10), *where the function* ψ_r *is given by definition* (3.6). *Next let the error potential* e_r *be given by formula* (4.5) *with* ψ_r *in place of* ψ *and let the constant* β *satisfy assumption* (2.2) *of Theorem* 2.1. *Finally, let the conclusions of Lemma* 8.1 *hold. Then*

$$\limsup_{\nu\to\infty} \sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} |e_r(\rho)|\,|\mathrm{Re}\,\theta_r(\rho)|^{-1}\,d\rho < \infty. \tag{8.15}$$

We start the proof of Theorem 8.2 with a technical proposition. In it we estimate the integral of the absolute value of $(\lambda - \psi_r)^{-1/2}$ times each of the five terms of the algebraic Lemma 6.3. This technical proposition is a version of Proposition 6.4, in which we have the complements $\mathscr{J}_r\setminus\mathscr{J}_{m2}$ in place of the intervals $\mathscr{J}_r$ on the left side and greater exponents of ν in place of the exponents of Proposition 6.4 on the right side.

PROPOSITION 8.3. *Let the assumptions and notations of Theorem* 8.2 *hold and instead of assumption* (2.2) *of Theorem* 2.1 *let the more general assumption* $0 \le \beta \le 1$ *hold. Then,*

$$\nu^{-2}\cdot \sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{-1/2}\psi_\nu\right|(\rho)\,d\rho = O(\nu^{-1}), \qquad \text{for } \nu\to\infty. \tag{8.16}$$

and

$$\nu^{-2}\cdot \sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{-3/2}\psi_\nu^2\right|(\rho)\,d\rho = O(\nu^{-\beta}), \qquad \text{for } \nu\to\infty. \tag{8.17}$$

Furthermore

$$\sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{-5/2}\psi_r'^2\right|(\rho)\,d\rho = O(\nu^{3-4\beta}), \qquad \text{for } \nu\to\infty, \tag{8.18}$$

$$\sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{-3/2} a_r\left[\left(p_0'' + b^2 p_0\right) + 4(a_r - 1)p_0^2\right]\right|(\rho)\,d\rho$$
$$= O(\nu^{3-4\beta}), \tag{8.19}$$

and

$$\sup_{\lambda\in\mathscr{J}} \int_{\mathscr{J}_r\setminus\mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{-3/2}[2a_r'p_0' + a_r''p_0]\right|(\rho)\,d\rho = O(\nu^{3-4\beta}),$$
$$\text{for } \nu\to\infty. \tag{8.20}$$

We prove Proposition 8.3 similarly to the proof of Proposition 6.4. In the following lemma we show that Lemma 8.1 allows us to adapt Lemma 6.5 to the complements $\mathscr{J}_r \setminus \mathscr{J}_{m2}$.

LEMMA 8.4. *Let the assumptions and notations of Proposition* 8.3 *hold and let the given real numbers* ω_2, ω_3 *satisfy*

$$\omega_2 > 1/2, \qquad \omega_3 \geq 0. \tag{8.21}$$

Then, for each $\omega_1 \in \mathscr{R}^+$ *as* $\nu \to \infty$,

$$\sup_{\lambda \in \mathscr{J}} \int_{\mathscr{J}_r \setminus \mathscr{J}_{m2}} \left|(\lambda - \psi_r)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right|(\rho)\, d\rho$$
$$= \begin{cases} O\left(\nu^{\max\{1+(\omega_1+\omega_3+1)\cdot(2\beta-2),\,1\}+\omega_3(1-\beta)}\right), & \omega_1+\omega_3 \neq -1 \\ O\left(\nu^{1+\omega_3(1-\beta)} \cdot \left(1 + |\log \nu^{2-2\beta}|\right)\right), & \omega_1+\omega_3 = -1. \end{cases} \tag{8.22}$$

To prove Lemma 8.4 we employ the factorization

$$(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} = (\lambda - \psi_\nu)^{\omega_1+\omega_3} \psi_\nu^{\omega_2} \cdot (\lambda - \psi_\nu)^{-\omega_3} a_r^{\omega_3}.$$

Since according to assumption (8.21) $\omega_3 \geq 0$, we see from conclusion (8.6) of Lemma 8.1 that the supremum of the absolute value of the second factor is of the order of $\nu^{\omega_3(1-\beta)}$. Hence,

$$\left|(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right| = O(\nu^{\omega_3(1-\beta)}) \cdot (\lambda - \psi_\nu)^{\omega_1+\omega_3} \psi_\nu^{\omega_2}.$$

Next we integrate the previous estimate over the complements $\mathscr{J}_r \setminus \mathscr{J}_{m2}$, which yields

$$\int_{\mathscr{J}_r \setminus \mathscr{J}_{m2}} \left|(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right|(\rho)\, d\pi$$
$$= O(\nu^{\omega_3(1-\beta)}) \cdot \int_{\mathscr{J}_r \setminus \mathscr{J}_{m2}} (\lambda - \psi_\nu)^{\omega_1+\omega_3} \psi_\nu^{\omega_2}\big|(\rho)\, d\rho. \tag{8.23}$$

According to assumption (3.7),

$$\lambda^{1/2}\nu^{-1} \cdot \mathscr{J}_r \subset \left(1 + \nu^{2\beta-2}, \infty\right),$$

and so, conclusion (5.1) of the scaling Lemma 5.3 with $\omega_1 + \omega_3$ in place of ω_1 and with the complements $\mathscr{J}_r \setminus \mathscr{J}_{m2}$ in place of $\mathscr{J}$ gives

$$\int_{\mathscr{J}_r \setminus \mathscr{J}_{m2}} \left|(\lambda - \psi_\nu)^{\omega_1+\omega_3} \psi_\nu^{\omega_2}\right|(\rho)\, d\rho$$
$$\leq \nu^{\omega_3(1-\beta)} \cdot \lambda^{\omega_1+\omega_2+\omega_3-1/2} \int_{1+\nu^{2\beta-2}}^{\infty} |\sigma^2 - 1|^{\omega_1+\omega_3} \sigma^{-2(\omega_1+\omega_3+\omega_2)}\, d\sigma.$$

Note that the improper integral on the right does exist, since by assumption (8.21) $-2\omega_2 < -1$. Finally, estimating this improper integral, using estimate (8.23) and conclusion (8.7) of Lemma 8.1, we obtain conclusion (8.22). This completes the proof of Lemma 8.4.

We complete the proof of Proposition 8.3 by repeated applications of Lemma 8.4. To prove conclusion (8.16) we note that conclusion (8.22) of Lemma 8.4 yields

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r} \left| (\lambda - \psi_r)^{\omega_1} \psi_\nu)^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho = O(\nu^{1+\omega_3(1-\beta)}),$$

$$\omega_1 + \omega_3 \geq -\frac{1}{2},\ \omega_2 > \frac{1}{2},\ \nu \to \infty. \tag{8.24}$$

Applying estimate (8.24) to $\omega_1 = -1/2$, $\omega_2 = 1$, $\omega_3 = 0$, we find conclusion (8.16).

To prove conclusion (8.17) we note that conclusion (8.22) of Lemma 8.4 yields

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho = O(\nu^{2-\beta+\omega_3(1-\beta)}),$$

$$\omega_1 + \omega_3 \geq -\frac{3}{2},\ \omega_2 > \frac{1}{2},\ \nu \to \infty. \tag{8.25}$$

Applying estimate (8.25) to $\omega_1 = -3/2$, $\omega_2 = 1$, $\omega_3 = 0$, we find conclusion (8.17).

To prove conclusion (8.18) we take the square of inequality (7.22). Then using the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$ we find

$$\psi_r'^2 \leq 8\nu^{-2}\psi_\nu^3 + 4a_r^2 p_0'^2 + 4a_r'^2 p_0^2. \tag{8.26}$$

To estimate ν^2 times the integral corresponding to the first term on right of inequality (8.26) we note that conclusion (8.22) of Lemma 8.4 yields

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho = O(\nu^{4-3\beta+\omega_3(1-\beta)}),$$

$$\omega_1 + \omega_3 \geq -\frac{5}{2},\ \omega_2 > \frac{1}{2},\ \nu \to \infty. \tag{8.27}$$

To estimate the integral corresponding to the second term on right of inequality (8.26) we apply estimate (8.24) to $\omega_1 = -5/2$, $\omega_2 = \beta$, $\omega_3 = 2$.

Then using estimate (2.15) we find

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_r)^{-5/2} a_r^2 p_0'^2 \right| (\rho)\, d\rho = O(\nu^{3-4\beta}), \qquad \text{for } \nu \to \infty. \tag{8.28}$$

To estimate the integral corresponding to the third term on right of inequality (8.26) we combine formula (6.29) with estimate (2.15). Then using conclusion (8.7) of Lemma 8.1 we find

$$\begin{aligned} &\int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_r)^{-5/2} a_r'^2 p_0^2 \right| (\rho)\, d\rho \\ &\qquad = O(\nu^{-2-2\beta}) \cdot \sup_{\lambda \in \mathcal{I}} \int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_\nu)^{-5/2-4} \psi_\nu^{3+\beta} a_r^4 \right| (\rho)\, d\rho. \end{aligned}$$

Application of estimate (8.27) to $\omega_1 = -5/2 - 4$, $\omega_2 = 3 + \beta$, $\omega_3 = 4$ shows that the second factor on the right is of the order of $\nu^{8+7\beta}$ and so,

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_r)^{-5/2} a_r'^2 p_0^2 \right| (\rho)\, d\rho = O(\nu^{3(2-3\beta)}), \qquad \text{for } \nu \to \infty. \tag{8.29}$$

Finally, we multiply inequality (8.26) by $|(\lambda - \psi_r)^{-5/2}|$ and integrate the resulting inequality over the complements $\mathcal{I}_r \setminus \mathcal{I}_{m2}$. Then using estimates (8.29), (8.28), and estimate (8.27) applied to $\omega_1 = -5/2$, $\omega_2 = 3$, $\omega_3 = 0$ we arrive at conclusion (8.18).

To prove conclusion (8.19) first we combine estimate (2.16) with an application of estimate (8.24) to $\omega_1 = -3/2$, $\omega_2 = (1 + \beta)/2$, $\omega_3 = 1$. Then we find

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_\nu)^{-3/2} a_r \big(p_0'' + b^2 p_0 \big) \right| (\rho)\, d\rho = O(\nu^{1-2\beta}), \qquad \text{for } \nu \to \infty. \tag{8.30}$$

To prove conclusion (8.19) second we show that

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{I}_r \setminus \mathcal{I}_{m2}} \left| (\lambda - \psi_r)^{-3/2} a_r (a_r - 1) p_0^2 \right| (\rho)\, d\rho = O(\nu^{3-4\beta}), \quad \text{for } \nu \to \infty. \tag{8.31}$$

Indeed, combining formula (6.29) with conclusion (8.7) of Lemma 8.1 and with estimate (2.15) we see that

$$\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_r)^{-3/2} a_r(a_r-1)p_0^2\right|(\rho)\,d\rho$$

$$= O(\nu^{-2\beta})\cdot\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_\nu)^{-3/2-1}\psi_\nu^{\beta}a_r^2\right|(\rho)\,d\rho. \quad (8.32)$$

Application of estimate (8.24) to $\omega_1=-3/2-1$, $\omega_2=\beta$, $\omega_3=2$ shows that the second factor on the right is of the order of $\nu^{3-2\beta}$ and so, estimate (8.31) follows. Now combining estimates (8.31) and (8.30) we arrive at conclusion (8.19).

To prove conclusion (8.20) first we show that

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_r)^{-3/2}|a_r'p_0'\right|(\rho)\,d\rho = O(\nu^{3-4\beta}), \qquad \text{for } \nu\to\infty. \quad (8.33)$$

Indeed, combining formula (6.29) with estimate (2.15) and using conclusion (8.7) of Lemma 8.1 we find

$$\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_r)^{-3/2}|a_r'p_0'\right|$$

$$= O(\nu^{-1-\beta})\cdot\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_r)^{-3/2-2}\psi_\nu^{(3+\beta)/2}a_r^2\right|(\rho)\,d\rho. \quad (8.34)$$

Applying estimate (8.25) to $\omega_1=-3/2-2$, $\omega_2=(3+\beta)/2$, $\omega_3=2$ we find that the second factor is of the order of $\nu^{4-3\beta}$ and so, estimate (8.33) follows. To prove conclusion (8.20) second we show that

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda-\psi_r)^{-3/2}a_r''p_0\right|(\rho)\,d\rho = O(\nu^{3-5\beta}), \qquad \text{for } \nu\to\infty. \quad (8.35)$$

To see estimate (8.35) we combine formula (6.35) with conclusion (8.7) of Lemma 8.1 and with estimate (2.15). This gives

$$\int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{-3/2} a_r'' p_0 \right| (\rho)\, d\rho$$

$$= O(\nu^{-2-\beta}) \cdot \sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{-3/2-2} \psi_\nu^{2+\beta/2} a_r^2 \right| (\rho)\, d\rho$$

$$+ O(\nu^{-2-\beta}) \cdot \sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{-3/2-3} \psi_\nu^{3+\beta/2} a_r^3 \right| (\rho)\, d\rho,$$

$$\text{for } \nu \to \infty. \quad (8.36)$$

Application of estimate (8.25) to $\omega_1 = -3/2 - 2$, $\omega_2 = 2 + \beta/2$, $\omega_3 = 2$ shows that the second factor of the first term on the right of estimate (8.36) is of the order of $\nu^{4-3\beta}$. Another application of estimate (8.25) to $\omega_1 = -3/2 - 3$, $\omega_2 = 2 + \beta/2$, $\omega_3 = 3$ shows that the second factor of the second term on the right of estimate (8.36) is of the order of $\nu^{5-4\beta}$. By assumption $1 - \beta \geq 0$ and so, $\nu^{3-5\beta}$dominates $\nu^{2-4\beta}$. That is to say the second term on the right of estimate (8.36) dominates the first one and so, estimate (8.35) follows. Finally, combining estimates (8.33) and (8.35) and using that by assumption $\beta \geq 0$, we arrive at conclusion (8.20). This completes the proof of Proposition 8.3.

Incidentally, note that if instead of assumption (2.2) of Theorem 2.1 we make the more restricted assumption $3/4 \leq \beta \leq 1$ then, Proposition 8.3 and the algebraic Lemma 6.3 imply Theorem 8.2.

We continue the proof of Theorem 8.2 by formulating sharper versions of the last three conclusions of Proposition 8.3.

PROPOSITION 8.5. *Let the assumptions and notations of Proposition* 8.3 *hold*. *Then*,

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{-3/2} a_r \cdot \left[\left(p_0'' + b^2 p_0 \right) + 4(a_r - 1) p_0^2 \right] \right| (\rho)\, d\rho$$

$$= O(\nu^{2-3\beta}). \quad (8.37)$$

Furthermore,

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left| (\lambda - \psi_r)^{-5/2} \psi_r'^2 \right| (\rho)\, d\rho = O(\nu^{2-3\beta}) + O(\nu^{3(2-3\beta)}),$$

$$\textit{for } \nu \to \infty \quad (8.38)$$

and

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}}\left|(\lambda-\psi_r)^{-3/2}[2a_r'p_0'+a_r''p_0]\right|(\rho)\,d\rho=O(\nu^{2-3\beta}),\quad \textit{for } \nu\to\infty. \tag{8.39}$$

We prove Proposition 8.3 similarly to the proof of Proposition 8.3 by formulating a sharper version of Lemma 8.4.

LEMMA 8.6. *Let the assumptions and notations of Lemma* 8.4 *hold. Then, for each* $\omega_1\in\mathscr{R}^+$ *as* $\nu\to\infty$,

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}}\left|(\lambda-\psi_r)^{\omega_1}\psi_\nu^{\omega_2}a_r^{\omega_3}\right|(\rho)\,d\rho$$
$$=\begin{cases} O(\nu^{\max\{(\omega_1+\omega_3+1)\cdot(2\beta-2)+1,1\}})+O(\nu^{\max\{(\omega_3-1)\cdot(1-\beta)+1,1\}}), \\ \qquad\qquad \omega_1+\omega_3\neq-1,\ \omega_3\neq1 \\ O\big(\nu(1+|\log\nu^{2-2\beta}|)\big)+O(\nu^{\max\{(\omega_3-1)\cdot(1-\beta)+1,1\}}), \\ \qquad\qquad \omega_1+\omega_3=-1,\ \omega_3\neq1 \\ O\big(\nu(1+|\log\nu^{2-2\beta}|)\big), \\ \qquad\qquad \omega_1+\omega_3=-1,\ \omega_3=1. \end{cases} \tag{8.40}$$

Since the proof of Lemma 8.6 is more technical, we do it in the Appendix.

We complete the proof of Proposition 8.3 by repeated applications of Lemma 8.4. To prove conclusion (8.37) recall the proof of conclusion (8.19) of Proposition 8.3, which shows that conclusion (8.37) is implied by the following sharper version of estimate (8.31):

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}}\left|(\lambda-\psi_r)^{-3/2}a_r(a_r-1)p_0^2\right|(\rho)\,d\rho=O(\nu^{2-3\beta}),\quad\text{for }\nu\to\infty. \tag{8.41}$$

To see estimate (8.41) we apply Lemma 8.6 to $\omega_1=-3/2$, $\omega_2=\beta$, $\omega_3=2$, which yields

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}}\left|(\lambda-\psi_\nu)^{-3/2-1}\psi_\nu^{\beta}a_r^2\right|(\rho)\,d\rho=O(\nu^{2-\beta}). \tag{8.42}$$

Combining estimates (8.42) and (8.32) we find estimate (8.41) and so, conclusion (8.37) follows.

To prove conclusion (8.38) recall the proof of conclusion (8.18) of Proposition 8.3, which shows that conclusion (8.38) is implied by the following sharper version of estimate (8.28):

$$\int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} \left|(\lambda - \psi_r)^{-5/2} a_r^2 p_0'^2\right|(\rho)\, d\rho = O(\nu^{2-3\beta}), \qquad \text{for } \nu \to \infty.$$

Now the previous estimate is an immediate consequence of estimates (8.42) and (2.15) and so, conclusion (8.38) follows.

To prove conclusion (8.39) recall the proof of conclusion (8.20) of Proposition 8.3, which shows that conclusion (8.39) is implied by the following sharper version of estimate (8.33):

$$\sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_r \setminus \mathscr{I}_{m2}} |(\lambda - \psi_r)^{-3/2} |a_r' p_0'| = O(\nu^{1-2\beta}), \qquad \text{for } \nu \to \infty.$$

Now the previous estimate is an immediate consequence of estimates (8.42) and (8.34) and so, conclusion (8.39) follows. This completes the proof of Proposition 8.5. Then, combining Proposition 8.5 with the algebraic Lemma 6.3 we arrive at Theorem 8.2.

We conclude this section by showing that the approximate phase of definition (8.5) satisfies the assumptions of the general Proposition 4.1 over the interval of definition (8.4). We do this by showing that conclusion (5.2) of Theorem 5.1 holds for the intervals $\mathscr{I}_{m2}$ in place of the intervals $\mathscr{I}_{m1}$:

$$\limsup_{\nu \to \infty} \sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_{m2}} |e_m(\rho)| \left|\operatorname{Re} \theta_m(\rho)\right|^{-1} d\rho < \infty. \tag{8.43}$$

For brevity we prove estimate (8.43), by proving the following version of conclusion (5.7) of Proposition 5.2:

$$\limsup_{\nu \to \infty} \sup_{\lambda \in \mathscr{I}} \int_{\mathscr{I}_{m2}} \left|(\lambda - \psi_\nu)^{-1/2} p_0\right|(\rho)\, d\rho = O(1), \qquad \text{for } \nu \to \infty. \tag{8.44}$$

Similarly to the proof of conclusion (5.7), we prove estimate (8.44) with the help of the scaling Lemma 5.3. Indeed, according to definitions (8.4) and (8.3)

$$\lambda^{1/2} \nu^{-1} \cdot \mathscr{I}_{m2} = d_\lambda^{-1} \cdot \left(1 - \frac{1-d}{2} \nu^{\beta - 1}, 1 + 2\nu^{\beta-1}\right).$$

Hence, conclusion (5.10) of the scaling Lemma 5.3 with the intervals $\mathscr{I}_{m2}$ in place of the interval $\mathscr{I}$ yields

$$\int_{\mathscr{I}_{m2}} \left|(\lambda - \psi_\nu)^{-1/2} \psi_\nu^{\beta/2}\right|(\rho)\, d\rho$$

$$= \lambda^{\beta/2-1} \cdot \nu \cdot \int_{d_\lambda^{-1}\cdot(1-((1-d)/2)\nu^{\beta-1})}^{d_\lambda^{-1}\cdot(1+2\nu^{\beta-1})} |\sigma^2 - 1|^{\omega_1} \sigma^{-2(\omega_1+\omega_2)}\, d\sigma. \tag{8.45}$$

Another application of definition (8.3) shows that the integrand on the right of formula (8.45) is bounded and that the lengths of these intervals are of the order of $\nu^{\beta-1}$. Hence the entire right side is of the order of ν^{β}. This fact together with estimate (2.15) gives estimate (8.45). Thus, estimate (8.44) follows and so, the function of definition (8.5) satisfies the approximate phase assumptions on the intervals $\mathscr{I}_{m2}$.

9. THE PROOF OF THEOREM 3.1 WITHOUT THE ADDITIONAL ASSUMPTION (3.4)

In this section we show that conclusion (3.12) of Theorem 3.1 holds without the additional assumption (3.4). For this purpose we define a partition of the intervals $\mathscr{I}_r$,

$$\mathscr{I}_{r1} = (\inf \mathscr{I}_r, \inf \mathscr{I}_{m2}), \qquad \mathscr{I}_{r2} = (\sup \mathscr{I}_{m2}, \infty), \tag{9.1}$$

so that, for large enough ν,

$$\mathscr{I}_r = \mathscr{I}_{r1} \cup \mathscr{I}_{r2} \cup \mathscr{I}_{r2}, \qquad \mathscr{I}_{r1} \cap \mathscr{I}_{r2} = \varnothing, \qquad \mathscr{I}_{r1} \cap \mathscr{I}_{m2} = \varnothing,$$
$$\mathscr{I}_{r2} \cap \mathscr{I}_{m2} = \varnothing. \tag{9.2}$$

Then we show that conclusion (7.6) of Proposition 7.2 holds with the intervals $\mathscr{I}_{r1}$ in place of the intervals $\mathscr{I}_r$ and with $\sup \mathscr{I}_{r1}$ in place of ρ:

$$F(\theta_r, f)(\sup \mathscr{I}_{r1}) \geq \gamma \cdot \nu^{1-\beta} \cdot F(\theta_r, f)(\inf \mathscr{I}_{r1}), \qquad \text{for } \nu > \nu_0. \tag{9.3}$$

According to Section 8 the function θ_r of definition (3.10) satisfies the approximate phase assumptions over the intervals $\mathscr{I}_{r1}$. Then, conclusion (4.9) of the general Proposition 4.1 with $\mathscr{I}_{r1}$ in place of $\mathscr{I}$ and with θ_r in place of θ yields

$$F(\theta_r, f)(\sup \mathscr{I}_{r1}) \geq \gamma \cdot \exp\left[2\int_{\mathscr{I}_{r1}} \operatorname{Im} \theta_r(\sigma)\, d\sigma\right] \cdot F(\theta_r, f)(\inf \mathscr{I}_{r1}),$$
$$\text{for } \nu > \nu_0. \tag{9.4}$$

Next we show that

$$\exp\left[2\int_{\mathscr{I}_{r1}} \operatorname{Im}\,\theta_r(\sigma)\,d\sigma\right] > \gamma\nu^{1-\beta}, \qquad \text{for } \nu > \nu_0. \tag{9.5}$$

Indeed combining conclusions (8.7) and (8.6) of Lemma 8.1 we find that similarly to estimate (7.20),

$$\left|\left[(\lambda - \psi_r)^{-1} a_r p_0\right](\partial\mathscr{I}_r)\right| = O(\nu^{1-2\beta}), \qquad \text{for } \nu \to \infty. \tag{9.6}$$

Next we show that

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda - \psi_r)^{-2}\psi_r' a_r p_0\right|(\rho)\,d\rho = O\Big(\nu^{2-3\beta}\big(\nu(1 + |\log \nu^{2-2\beta}|)\big)\Big),$$
$$\text{for } \nu \to \infty. \tag{9.7}$$

To see estimate (9.7) we repeat the proof of estimate (7.21) with the estimate

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_r\setminus\mathscr{I}_{m2}} \left|(\lambda - \psi_\nu)^{\omega_1}\psi_\nu^{\omega_2} a_r^{\omega_3}\right|(\rho)\,d\rho$$
$$= \begin{cases} O\left(\nu^{\max\{(\omega_3-1)\cdot(1-\beta)+1,1\}}\right), & \omega_3 \neq 1 \\ O\big(\nu(1 + |\log \nu^{2-2\beta}|)\big), & \omega_3 = 1 \end{cases} \tag{9.8}$$

Note that the right side of estimate (9.8) is the second term of conclusion (8.40) of Lemma 8.6. We omitted the first term since this term leads to estimate (7.17) and we have seen in the proof of estimate (7.21) that it gives terms for which estimate (9.7) holds. Next we apply estimate (9.8) to $\omega_1 = -2$, $\omega_3 = 2$, $\omega_3 = 3$, and to $\omega_1 = -2$, $\omega_2 = \beta$, $\omega_3 = 2$. Then we obtain weaker versions of estimates (7.25), (7.24), and (7.23). More specifically, we obtain estimates where the intervals $\mathscr{I}_r$ on the left are replaced by the complements and the order $\nu^{1-2\beta}$ on the right is replaced by the order $\nu^{2-3\beta}$. Similarly to the way that estimates (7.25), (7.24), and (7.23) imply estimate (7.21) we see that these weaker versions imply estimate (9.7).

Similarly to the way that estimates (7.21) and (7.20) imply lower estimate (7.8) we see that estimates (9.7) and (9.6) imply the lower estimate

$$\exp\left[2\int_{\inf\mathscr{I}_{r1}}^{\rho} \operatorname{Im}\,\theta_r(\sigma)\,d\sigma\right] > \gamma\nu^{1-\beta}, \qquad \text{for } \nu > \nu_0 \text{ and } \rho \geq \delta\lambda^{-1/2}\nu. \tag{9.9}$$

We see from definition (9.1) and from relation (8.47) that

$$\sup \mathscr{J}_{r1} = \inf \mathscr{J}_{m2} \geq \lambda^{-1/2}\delta\nu, \qquad \text{with } \delta = \left(\frac{1}{2} + \frac{1}{2d}\right) > 1, \quad (9.10)$$

and so, we can choose $\rho = \delta$ in the lower estimate (9.9). Hence the lower estimate (9.5) follows.

We have also seen in Section 8 that function θ_m of definition (3.6) satisfies the assumptions of the general Proposition 4.1 over the intervals $\mathscr{J}_{m2}$. Hence conclusion (4.9) holds with $\mathscr{J}_{m2}$ in place of $\mathscr{J}$ and the function θ_m in place of θ. Then combining this conclusion with the fact that according to formula (7.4) Im $\theta_m > 0$, we find

$$F(\theta_m, f)(\sup \mathscr{J}_{m2}) \geq \gamma \cdot F(\theta_m, f)(\inf \mathscr{J}_{m2}) \qquad \text{for } \nu > \nu_0. \quad (9.11)$$

We see that similarly to estimate (9.11),

$$F(\theta_r, f)(\rho) \geq \gamma \cdot F(\theta_r, f)(\inf \mathscr{J}_{m2}) \qquad \text{for } \nu > \nu_0,\ \rho \in \mathscr{J}_{r2}.$$

Thus the lower estimate (7.33) holds under the additional assumption (8.1) and so, conclusion (3.12) follows. That is to say we have removed the additional assumption (3.4) in the proof of Theorem 3.1.

APPENDIX. PROOF OF LEMMA 8.6

For brevity we prove Lemma 8.6 in the general case of $\omega_1 + \omega_3 \neq -1$ and $\omega_3 \neq 1$ only. The exceptional cases follow by minor adjustments. We start by showing that on one of the intervals of the partition (9.2) the absolute value of one of the factors of conclusion (8.40) remains bounded. More specifically,

$$\limsup_{\nu \to \infty} \sup_{\lambda \in \mathscr{I}} \sup_{\rho \in \mathscr{J}_{r2}} \left|(\lambda - \psi_\nu)^{\omega_1}\right|(\rho) < \infty, \qquad \omega_1 \in \mathscr{R}. \qquad (\text{A.1})$$

We see from definitions (9.1) and (2.14) and from assumption (3.7) that estimate (A.1) is implied by

$$\sup_{\lambda \in \mathscr{I}} \sup_{\rho \in \mathscr{J}_{r2}} \left|(\lambda - \psi_\nu(\rho))^{-1}\right| < \infty. \qquad (\text{A.2})$$

To prove estimate (A.2), we apply formula (7.29) with $\rho = \inf \mathscr{J}_{r2}$. Then using definitions (9.1), (8.4), and (8.3) we find

$$\lambda - \psi_\nu(\inf \mathscr{J}_{r2}) = \left[d_\lambda^{-1}(1 + 2\nu^{\beta-1}) - 1\right] \cdot \lambda_b (1 + 2\nu^{\beta-1})^{-1} \times \left[d_\lambda^{-1} + (1 + 2\nu^{\beta-1})^{-1}\right]. \qquad (\text{A.3})$$

We see from definitions (8.3) and (8.2) and from the additional assumption (8.1) that the right side of formula (A.3) is strictly positive. In fact, it is bounded away from 0:

$$\liminf_{\nu\to\infty} \inf_{\lambda\in\mathscr{I}} (\lambda - \psi_\nu)^{-1}(\inf \mathscr{I}_{r2}) > 0.$$

Since the function $(\lambda - \psi_\nu)$ is monotone the previous estimate and relation (2.16) together give estimate (A.2). Hence estimate (A.1) follows.

We continue the proof of conclusion (8.40) by showing that it holds for the intervals $\mathscr{I}_{r2}$ in place of the complements $\mathscr{I}_r \setminus \mathscr{I}_{m2}$. This is implied by

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_{r2}} \left|(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right|(\rho)\, d\rho = O(\nu^{\max\{(-\omega_3+1)\cdot(\beta-1)+1,1\}}). \quad \text{(A.4)}$$

To prove estimate (A.4) note that formula (8.8) yields

$$(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} = (\lambda - \psi_\nu)^{\omega_1+\omega_3} \psi_\nu^{\omega_2}(\lambda_b - \psi_\nu)^{-\omega_3}.$$

Hence estimate (A.1) with $\omega_1 + \omega_3$ in place of ω_1 gives

$$\left|(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right| = O(1) \cdot \left|\psi_\nu^{\omega_2}(\lambda_b - \psi_\nu)^{-\omega_3}\right|. \quad \text{(A.5)}$$

Integrating estimate (A.5) over the intervals $\mathscr{I}_{r2}$ we find

$$\sup_{\lambda\in\mathscr{I}} \int_{\mathscr{I}_{r2}} \left|(\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3}\right|(\rho)\, d\rho$$
$$= O(1) \cdot \int_{\mathscr{I}_{r2}} \left|\psi_\nu^{\omega_2}(\lambda_b - \psi_\nu)^{-\omega_3}\right|(\rho)\, d\rho.$$

According to definitions (9.1) and (8.4),

$$\lambda_b^{1/2}\nu^{-1} \cdot \mathscr{I}_{r2} = (1 + 2\nu^{\beta-1}, \infty).$$

Hence, conclusion (5.10) of the scaling Lemma 5.3 for the intervals $\mathscr{I}_{r2}$ in place of $\mathscr{I}$, with λ_b in place of λ and with $-\omega_3$ in place of ω_1 yields

$$\int_{\mathscr{I}_{r2}} \left|\psi_\nu^{\omega_2}(\lambda_b - \psi_\nu)^{-\omega_3}\right|(\rho)\, d\rho$$
$$= \lambda_b^{\omega_3+\omega_2-1/2} \cdot \nu \cdot \int_{1+2\nu^{\beta-1}}^{\infty} |\sigma^2 - 1|^{-\omega_3} \sigma^{-2(-\omega_3+\omega_2)}\, d\sigma. \quad \text{(A.6)}$$

Estimating the elementary integral on the right of estimate (A.6) and using estimate (A.5) we obtain estimate (A.4).

We complete the proof of conclusion (8.40) by showing that it holds for the interval $\mathcal{J}_{r1}$ in place of the complement $\mathcal{J}_r \setminus \mathcal{J}_{m2}$: This is implied by

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{J}_{r1}} \left| (\lambda - \psi_\nu)^{\omega_1} \cdot \psi_\nu^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho$$
$$= O(\nu^{\max\{(\omega_1+1)\cdot(2\beta-2)+1,1\}}) + O(\nu^{\max\{(\omega_3+1)\cdot(\beta-1)+1,1\}}). \quad \text{(A.7)}$$

To prove estimate (A.7), with the help of definitions (8.3) and (8.2) we define a partition of this interval. More specifically, we define

$$\mathcal{J}_{r11} = \left(\inf \mathcal{J}_r, \lambda_b^{-1/2}\nu(1 + d_\lambda - d)\right),$$
$$\mathcal{J}_{r12} = \left(\lambda_b^{-1/2}\nu(1 + d_\lambda - d), \inf \mathcal{J}_{m2}\right), \qquad \beta < 1, \quad \text{(A.8)}$$

so that for large enough ν, definitions (9.1), (8.4), and (8.3) and assumption (3.8) yield

$$\mathcal{J}_{r1} = \mathcal{J}_{r11} \cup \mathcal{J}_{r12}, \qquad \mathcal{J}_{r11} \cap \mathcal{J}_{r12} = \varnothing. \quad \text{(A.9)}$$

We claim that estimate (A.7) holds for the interval $\mathcal{J}_{r11}$ in place of the interval $\mathcal{J}_{r1}$, which is implied by

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{J}_{r11}} \left| (\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho = O(\nu^{\max\{(\omega_1+\omega_3+1)\cdot(2\beta-2)+1,1\}}).$$
$$\text{(A.10)}$$

The proof of Lemma 8.4 and formula (8.8) together show that to prove estimate (A.10) it suffices to prove that

$$\limsup_{\nu \to \infty} \sup_{\lambda \in \mathcal{I}} \sup_{\rho \in \mathcal{J}_{r11}} \left| (\lambda_b - \psi_\nu(\rho))^{-1} \right| < \infty. \quad \text{(A.11)}$$

To see estimate (A.11) we apply formula (7.29) with λ_b in place of λ and $\sup \mathcal{J}_{r11}$ in place of ρ. Then using definition (A.8), we find

$$\lambda_b - \psi_\nu(\sup \mathcal{J}_{r11}) = [d_\lambda - d] \cdot \lambda_b (1 + d_\lambda - d)^{-1} \left[1 + (1 + d_\lambda - d)^{-1}\right].$$
$$\text{(A.12)}$$

Similarly to the way that formula (8.11) implies estimate (8.10), we see that formula (A.12) implies estimate (A.11). Next we claim that estimate (A.7) also holds with the interval $\mathcal{J}_{r12}$ in place of the interval $\mathcal{J}_{r1}$ on the left and with the second term on the right:

$$\sup_{\lambda \in \mathcal{I}} \int_{\mathcal{J}_{r12}} \left| (\lambda - \psi_\nu)^{\omega_1} \psi_\nu^{\omega_2} a_r^{\omega_3} \right| (\rho)\, d\rho = O(\nu^{\max\{(-\omega_3+1)\cdot(\beta-1)+1,1\}}).$$
$$\text{(A.13)}$$

To prove estimate (A.13), first we show that estimate (A.3) holds with the intervals $\mathscr{I}_{r12}$, in place of the intervals $\mathscr{I}_{r2}$:

$$\limsup_{\nu\to\infty} \sup_{\lambda\in\mathscr{I}} \sup_{\rho\in\mathscr{I}_{r12}} \left|(\lambda-\psi_\nu(\rho))^{-1}\right| < \infty. \tag{A.14}$$

To see estimate (A.14) we apply formula (7.29) with $\inf \mathscr{I}_{r12}$ in place of ρ. Then using the definitions (A.8) and (8.3), we find

$$\begin{aligned}\lambda - \psi_\nu(\inf \mathscr{I}_{r12}) \\ &= \left[d_\lambda^{-1}(1-d)\right]\cdot\lambda_b(1+d_\lambda-d)^{-1}\left[d_\lambda^{-1}+(1+d_\lambda-d)^{-1}\right].\end{aligned}$$

Similarly to the way that formula (8.11) implies estimate (8.10) we see that the previous formula implies estimate (A.14). To prove estimate (A.13), second we show that

$$\sup_{\lambda\in\mathscr{I}}\int_{\mathscr{I}_{r12}} \left|\psi_\nu^{\omega_2}(\lambda_b-\psi_\nu)^{-\omega_3}\right|(\rho)\,d\rho = O(\nu^{\max\{(\omega_3+1)\cdot(\beta-1)+1,1\}}),$$

$$\text{for } \omega_3 \neq -1,\ \omega_2 > 1/2. \tag{A.15}$$

Indeed, we see from definitions (A.8) and (8.4) that

$$\lambda_b^{1/2}\nu^{-1}\cdot\mathscr{I}_{r12} = \left(1+d_\lambda-d, 1-\frac{1-d}{2}\nu^{\beta-1}\right).$$

Hence, conclusion (5.10) of the scaling Lemma 5.3 for the intervals $\mathscr{I}_{r12}$ in place of $\mathscr{I}$, with λ_b in place of λ and with $-\omega_3$ in place of ω_1 yields

$$\begin{aligned}\int_{\mathscr{I}_{r12}} \left|(\lambda_b-\psi_\nu)^{-\omega_3}\psi_\nu^{\omega_2}\right|(\rho)\,d\rho \\ &= \lambda_b^{-\omega_3+\omega_2-1/2}\cdot\nu\cdot\int_{1+d_\lambda-d}^{1-((1-d)/2)\nu^{\beta-1}} |\sigma^2-1|^{\omega_3}\sigma^{-2(\omega_3+\omega_2)}\,d\sigma.\end{aligned}$$

Using definition (8.4) to estimate the elementary integral on the right, we get estimate (A.15). Inserting estimate (A.15) into estimate (A.14) we find estimate (A.13). Combining estimates (A.13) and (A.10) with definitions (A.8) and (A.4) we obtain estimate (A.7). Finally, combining estimates (A.4) and (A.7) with definition (9.1) and with conclusion (8.7) of Lemma 8.1 we arrive at conclusion (8.40). This completes the proof of Lemma 8.6.

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